

SYMPLECTIC A-DIRECTED IMMERSIONS

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1. INTRODUCTION

Let V and W be smooth manifolds of dimensions n and q respectively with $n < q$ and let $\pi : Gr_n(W) \rightarrow W$ be the Grassmanian bundle of n -planes tangent to W , i.e $\pi^{-1}(x) = Gr_n(T_x W)$ for $x \in W$. For a given vector bundle monomorphism $F : TV \rightarrow TW$, define $GF : V \rightarrow Gr_n W$ given by

$$x \mapsto F_x(T_x V)$$

Now for an immersion $f : V \rightarrow W$ we call Gdf the tangential lift of f . Let $A \subset Gr_n(W)$ be an arbitrary subset then an immersion $f : V \rightarrow W$ is called A-directed if $Gdf(V) \subset A$. In [3] Gromov proved the following theorem using Convex Integration technique which is known as A-directed embedding.

Theorem 1.1. (*A-directed embedding*) *Let V and W be as above with V an open manifold. Let $A \subset Gr_n(W)$ be an open set and $f_0 : V \hookrightarrow W$ be an embedding such that the tangential lift $G_0 = Gdf_0$ is homotopic to a map $G_1 : V \rightarrow A \subset Gr_n(W)$ through a homotopy $G_t : V \rightarrow Gr_n(W)$ such that $\pi \circ G_t = f_0$, then f_0 can be isotoped to an A-directed embedding $f_1 : V \rightarrow W$. Moreover given a core $K \subset V$, the isotopy f_t can be chosen arbitrarily C^0 -close to f_0 on $Op(K)$.*

Alternative proof of the theorem has been provided in [2] and [4]. In this paper we have presented a proof of a symplectic analogue of this theorem. For V closed 1.1 is not true in general but under a completeness condition on A , 1.1 becomes true, see [2] for a proof.

Let us conclude this section by stating the main theorem of this paper. Let (V, ω_V) and (W, ω_W) be symplectic manifolds of dimensions $2n$ and $2q$ respectively with $n < q$. Let $Sp_{2n}(W) \subset Gr_{2n}(W)$ consists of symplectic $2n$ -planes tangential to W , and $\pi : Sp_{2n}(W) \rightarrow W$ be the bundle.

Theorem 1.2. *Let $(V, \omega_V = d\alpha_V)$ be an open exact symplectic manifold and $(W, \omega_W = d\alpha_W)$ be an exact symplectic manifold of dimensions $2n$ and $2q$ respectively with $n < q$. Let $A \subset Sp_{2n}(W)$ be an open subset and let $f_0 : V \rightarrow W$ be a symplectic immersion i.e, f_0 is an immersion satisfying $f_0^*(\omega_W) = \omega_V$. Further assume that the tangential lift $G_0 = Gdf_0 : V \rightarrow Sp_{2n}(W)$ admits a homotopy $G_t : V \rightarrow Sp_{2n}(W)$ such that $\pi \circ G_t = f_0$ and $G_1 : V \rightarrow A \subset Sp_{2n}(W)$ then f_0 admits a homotopy of isosymplectic immersions f_t such that f_1 is A-directed. Moreover given a core $K \subset V$, the homotopy f_t can be chosen arbitrarily C^0 -close to f_0 on $Op(K)$.*

2. MAIN THEOREM

In this section we will present a proof of 1.2. First we need an approximation result which we state below.

Theorem 2.1. (*Main Approximation Lemma*) $(V, \omega_V = d\alpha_V)$ be an open exact symplectic manifold as above and let $K \subset V$ be a polyhedron of positive codimension and $G_t : V \rightarrow Sp_{2n}(W)$ be a symplectic tangential homotopy, where $(W, \omega_W = d\alpha_W)$ as above is exact symplectic, such that $G_0 = Gdf$ for some isosymplectic immersion $f : V \rightarrow W$ and $\pi \circ G_t = f$. Then for arbitrarily small $\delta, \varepsilon > 0$ there exists a δ -small symplectic diffeotopy $h^\tau : V \rightarrow V$, $\tau \in [0, 1]$ and a homotopy of symplectic immersions

$$\tilde{f}_t : Op(\tilde{K}) \rightarrow W$$

where $\tilde{K} = h^1(K)$ and $\tilde{f}_0 = (f_0)|_{Op_V \tilde{K}}$, such that the homotopy $Gd\tilde{f}_t$ is ε -close to the tangential homotopy $(G_t)|_{Op_V \tilde{K}}$.

We need a theorem from [2] for the proof of 2.1 which involves the following concepts.

Definition 2.2. (*Local Integrability*) A differential relation $\mathcal{R} \subset J^r(V, W)$ is called (parametrically) locally integrable if given a map $h : I^k \rightarrow V$, a family of sections

$$F_p : h(p) \rightarrow \mathcal{R}, \quad p \in I^k$$

and a family local holonomic extensions near ∂I^k

$$\tilde{F}_p : Op(h(p)) \rightarrow \mathcal{R}, \quad \tilde{F}_p(h(p)) = F_p(h(p)), \quad p \in Op(\partial I^k)$$

there exists a family of local holonomic extensions

$$\tilde{F}_p : Op(h(p)) \rightarrow \mathcal{R}, \quad \tilde{F}_p(h(p)) = F_p(h(p)), \quad p \in I^k$$

such that for $p \in Op(\partial I^k)$ these new extensions coincide with the original extensions over $Op(h(p))$.

Definition 2.3. (*Microflexibility*) Let $K^m = [-1, 1]^m$. For a fixed n and any $k < n$ we denote by θ_k the pair $(K^n, K^k \cup \partial K^n)$. Let V be an n -dimensional manifold. A pair $(A, B) \subset V$ is called a θ_k pair if (A, B) is diffeomorphic to the standard pair θ_k .

A differential relation $\mathcal{R} \subset J^r(V, W)$ is called (parametrically) k -microflexible if for any sufficiently small open ball $U \subset V$ and any families parametrized by $p \in I^m$ of

- θ_k -pairs $(A_p, B_p) \subset U$,
- holonomic sections $F_p^0 : Op(A_p) \rightarrow \mathcal{R}$ and
- holonomic homotopies $F_p^\tau : Op(B_p) \rightarrow \mathcal{R}$, $\tau \in [0, 1]$, of the sections F_p^0 over $Op(B_p)$ which are constant over $Op(\partial B_p)$ for all $p \in I^m$ and constant over $Op(B)$ for $p \in Op(\partial I^m)$

there exists a number $\sigma > 0$ and a family of holonomic homotopies

$$F_p^\tau : Op(A_p) \rightarrow \mathcal{R}, \quad \tau \in [0, \sigma],$$

which extend the family of homotopies

$$F_p^\tau : Op(B_p) \rightarrow \mathcal{R}, \quad \tau \in [0, \sigma],$$

and are constant over $Op(\partial A_p)$ for all $p \in I^m$ and constant over $Op(A)$ for $p \in Op(\partial I^m)$.

A differential relation $\mathcal{R} \subset J^r(V, W)$ is called *microflexible* if \mathcal{R} is k -microflexible for all $k = 0, 1, 2, \dots, n-1$, where $n = \dim(V)$.

Definition 2.4. [2] Let $\mathcal{U} \subset \text{Diff}(V)$ be a lie subgroup of the group of compactly supported diffeomorphisms of V , and \mathcal{A} be its lie algebra. We call \mathcal{U} , (\mathcal{A}) "capacious" if it satisfies the following conditions

- for any $v \in \mathcal{A}$, any compact set $A \subset V$ and its neighborhood $U \supset A$ there exists a vector field $\tilde{v}_{A,U} \in \mathcal{A}$ which is supported in U and which coincides with v on A .
- given any tangent hyperplane $\tau \subset T_x V$, $x \in V$, there exists a vector field $v \in \mathcal{A}$ which is transversal to τ .

Moreover the above conditions need to be satisfied parametrically with respect to a compact parameter space.

It has been mentioned in [2] that the relation of Legendrian immersions is locally integrable and microflexible. Let us prove this. We need a lemma from [2] to prove this.

Lemma 2.5. (Contact Stability) Let ξ_t , $t \in I$, be a family of contact structures on a neighborhood $Op(A) \subset M$ of a compact set $A \subset M$. Then there exists an isotopy of $\phi_t : Op(A) \rightarrow M$, fixed on A such that $\phi_t^* \xi_0 = \xi_t$, $t \in I$.

Lemma 2.6. Let M be any manifold of dimension n and (N, ξ) be a contact manifold with dimension $2n+1$, then \mathcal{R}_{Leg} , the relation of Legendrian immersions from M to N is locally integrable and microflexible.

Proof. To avoid notational complexity, we shall only prove the non-parametric versions.

Local Integrability: Take $F : T_x M \rightarrow \xi_y \subset T_y N$ be an injective linear map. So now take an immersion $f : Op(x) \rightarrow Op(y)$ such that $df_x = F$. Take a hyper-plane field ξ' on $Op(y)$ such that $df(TOp(x)) \subset \xi'$, $\xi' \cap \xi_y^\perp$ and $\xi'_y = \xi_y$. As $\xi'_y = \xi_y$, ξ' is also a contact structure on $Op(y)$. Let $\xi = \ker(\eta)$, and $\xi' = \ker(\eta')$, where η and η' are one forms on $Op(y)$. We can also assume $\eta_y = \eta'_y$. Let $\eta_t = (1-t)\eta + t\eta'$, $t \in I$. So $(\eta_t)_y = \eta_y$. So $\xi_t = \ker(\eta_t)$ is a homotopy of contact structures on $Op(y)$. So by 2.5, we get an isotopy $\phi_t : Op(y) \rightarrow Op(y)$, such that $(\phi_t)^* \xi_0 = \xi_t$. As $\xi_0 = \xi$ and $\xi_1 = \xi'$, we get $(\phi_1)^* \xi = \xi'$. So the required Legendrian immersion is $\phi_1 \circ f$.

Microflexibility: Let $U, (A, B), F^\tau$ be as in 2.3. As the relation of immersions is microflexible, we get a $\tilde{\sigma} > 0$ such that there exists a homotopy

$$\tilde{f}^\tau : Op(A) \rightarrow N, \quad \tau \in [0, \tilde{\sigma}]$$

which extends the homotopy

$$f^\tau : Op(B) \rightarrow N, \text{ where } F^\tau = df^\tau$$

Now take a homotopy ξ_τ of hyperplane fields, such that $d\tilde{f}^\tau(TOp(A)) \subset \xi_\tau$ and $\xi_0 = \xi$. As U is small, $\xi_\tau = \ker \eta_\tau$ for some homotopy of one forms η_τ . Define

$$P : [0, \tilde{\sigma}] \rightarrow \Omega^{2n+1}(N), \text{ as } P(\tau) = \eta_\tau \wedge (d\eta_\tau)^n$$

So $P(0)$ is a non-zero section in $\Omega^{2n+1}(N)$. So there exists a $\sigma \in (0, \tilde{\sigma}]$ such that η_τ are contact forms for $\tau \in [0, \sigma]$. Again by 2.5 we get an isotopy ϕ_τ such that $(\phi_\tau)^* \ker(\eta_0) = \ker(\eta_\tau)$. So set

$$f^\tau : Op(A) \rightarrow N$$

as $\phi_\tau \circ \tilde{f}^\tau$, for $\tau \in [0, \sigma]$. □

Theorem 2.7. ([2]) *Let $\mathcal{R} \subset X^{(r)}$ be a locally integrable, microflexible differential relation. $K \subset V$ be a polyhedron of positive codimension and $F_z : OpK \rightarrow \mathcal{R}$ be a family of sections parametrized by a cube I^m , $m = 0, 1, 2, \dots$. Suppose the sections F_z are holonomic for $z \in Op(\partial I^m)$. Then for arbitrarily small $\delta, \varepsilon > 0$ there exists a family of δ -small diffeotopies $h_z^\tau : V \rightarrow V$, $\tau \in I$, $z \in I^m$ and a family of holonomic sections $\tilde{F}_z : Op(h_z^1(K)) \rightarrow \mathcal{R}$, $z \in I^m$ such that*

- $h_z^\tau = id_V$ and $\tilde{F}_z = F_z$ for all $z \in Op(\partial I^m)$
- $dist(\tilde{F}_z(v), (F_z)|_{Op(h_z^1(K))}(v)) < \varepsilon$ for all $v \in Op(h_z^1(K))$

Moreover if \mathcal{R} is \mathcal{U} -invariant, where $\mathcal{U} \subset Diff(V)$ is a "capacious" subgroup, then h_z^τ can be taken from \mathcal{U} , i.e., we can assume $h_z^\tau \in \mathcal{U}$.

Proposition 2.8. *Let $K \subset V$ be a polyhedron of positive codimension and $F_t : TV \rightarrow TW$ be a homotopy of vector bundle monomorphisms covering an isosymplectic immersion f with $F_0 = df$, (f - isosymplectic) such that $F_t^* \omega_W = \omega_V$. Then for arbitrarily small $\delta, \varepsilon > 0$, there exists a δ -small symplectic diffeotopy $h^\tau : V \rightarrow V$ and a homotopy of isosymplectic immersions $\tilde{f}_t : Op(\tilde{K}) \rightarrow W$, where $\tilde{K} = h^1(K)$ and $\tilde{f}_0 = f|_{Op(\tilde{K})}$ such that the homotopy $Gd\tilde{f}_t : Op(\tilde{K}) \rightarrow Sp_{2n}(W)$ is ε -close to $(GF_t)|_{Op(\tilde{K})}$.*

Proof. Define F_t' as

$$F_t' := F_{2t}, \quad t \in [0, 1/2] \text{ and } F_t' := F_{2-2t}, \quad t \in [1/2, 1]$$

As $f : V \rightarrow W$ is an isosymplectic immersion, we consider the following symplectic vector bundle $E \rightarrow V$ whose fiber over a point $v \in V$ is the space $(df(T_v V))^{\perp \omega_W}$, the ω_W -dual to $df(T_v V) \subset T_{f(v)} W$. By (9.2.2) of [2] there exists a symplectic structure $\tilde{\omega}$ on a neighborhood $Op(V)$ of the zero section V of E such that $\tilde{\omega}|_V = \omega_V$. As both ω_V and ω_W are exact, $\tilde{\omega}$ can also be taken to be exact, so let $\tilde{\omega} = d\tilde{\alpha}$. Now F_t' can be

extended to a homotopy of vector bundle morphism $F_t'' : TOp(V) \rightarrow TW$ which fiber wise isomorphism, such that $(F_t'')^* \omega_W = \tilde{\omega}$. Moreover F_0'' and F_1'' can be chosen to be holonomic by "Symplectic Neighborhood Theorem" (9.3.2) of [2]. Now consider the symplectic manifold $Op(V) \times W$ with symplectic structure $\tilde{\omega} \oplus (-\omega_W)$. Define

$$F_t^0 : TOp(V) \rightarrow TOp(V) \times TW$$

$$X \mapsto (X, F_t''(X))$$

Observe that for each fixed $t \in I$, and $x \in Op(V)$, $Im(F_t^0)_x$ is an isotropic subspace.

Now take $s : I \rightarrow I$ such that $s(t) = 0$, for $t \in [0, \sigma]$, and $s(t) = 1$, for $t \in [1 - \sigma, 1]$, where $0 < \sigma < 1/2$ is small and $s|_{[\sigma, 1 - \sigma]} : [\sigma, 1 - \sigma] \rightarrow I$ is a homeomorphism. Let $s^{-1}(1/2) = t_0$. Define \tilde{F}_t as $\tilde{F}_t := F_{s(t)}^0$. Observe that \tilde{F}_t is holonomic for $t \in Op(\partial I)$.

Consider the contact manifold $Op(V) \times W \times \mathbb{R}$ with contact structure $ker(dz - \eta)$, where $\eta = \tilde{\alpha} \oplus (-\alpha_W)$ and z is the variable in \mathbb{R} . The homotopy of isotropic monomorphisms \tilde{F}_t lifts to a homotopy of isotropic monomorphisms

$$L(\tilde{F}_t) : TOp(V) \rightarrow T(Op(V) \times W \times \mathbb{R})$$

which in our case is a homotopy of Legendrian monomorphisms.

Now in view of 2.6 we can use 2.7. So by 2.7 we get, for given arbitrarily small $\delta, \varepsilon > 0$ there exists a family of δ -small diffeotopies $h_t^\tau : Op(V) \rightarrow Op(V)$, $\tau \in I$, $t \in I$ and a family of holonomic sections $\tilde{F}_t^{hol} : Op(h_t^1(K)) \rightarrow \mathcal{R}_{Leg}$, $t \in I$ such that

- $h_t^\tau = id_{Op(V)}$ and $\tilde{F}_t^{hol} = L(\tilde{F}_t)$ for all $t \in Op(\partial I)$
- $dist(\tilde{F}_t^{hol}(v), L(\tilde{F}_t)|_{Op(h_t^1(K))}(v)) < \varepsilon$ for all $v \in Op(h_t^1(K))$

Moreover, as $\mathcal{U} = Ham(Op(V))$, the identity component of the group of compactly supported hamiltonian diffeomorphisms of the symplectic manifold $(Op(V), d\tilde{\alpha})$ is "capacious" and \mathcal{R}_{Leg} here is invariant under the action of \mathcal{U} , we can take $h_t^\tau \in \mathcal{U}$.

Consider \tilde{F}_t^{hol} , for $t \in [0, t_0]$ to get a homotopy of Legendrian immersions which we denote by $f_t^{hol} : Op(h_t^1(K)) \rightarrow Op(V) \times W \times \mathbb{R}$, for $t \in [0, t_0]$. Let $\pi : Op(V) \times W \times \mathbb{R} \rightarrow Op(V) \times W$ be the projection on the first two factors. Then $\pi \circ f_t^{hol} : Op(h_t^1(K)) \rightarrow Op(V) \times W$ is a homotopy of exact Lagrangian immersions into $(Op(V) \times W, d\eta)$.

Set $\tilde{K} = h_{t_0}^1(K) \subset h_{t_0}^1(V)$. As $h_{t_0}^1$ is a diffeomorphism, we identify V with $h_{t_0}^1(V)$. Now take $\tilde{v} \in Op_V(\tilde{K})$. So there exists a unique $v \in Op_V(K)$ such that $\tilde{v} = h_{t_0}^1(v)$. Now consider the curve

$$\gamma_{\tilde{v}} : [0, t_0] \rightarrow Op(V) \times W, \quad \gamma_{\tilde{v}}(t) = \pi \circ f_{t_0-t}^{hol}(h_{t_0-t}^1(v))$$

Set $\tilde{f}'_t(\tilde{v}) = \gamma_{\tilde{v}}(t_0 - t)$, for $t \in [0, t_0]$. So $\tilde{f}'_t(\tilde{v}) = (\tilde{v}, \tilde{f}_t(\tilde{v}))$. As the restriction of a isotropic immersion is isotropic, so $(\tilde{f}_t)|_V$ is isosymplectic. Now the required homotopy of isosymplectic immersions is given by reparametrizing $(\tilde{f}_t)|_V$. \square

Proof. (Proof of 2.1) For $\bar{\varepsilon} \in (0, \pi/4)$ choose an integer N such that for each interval

$$\Delta_i = [(i-1)/N, i/N]$$

the homotopy $\{G_t\}_{t \in \Delta_i}$ is $\bar{\varepsilon}$ -small. Set $K_0 = K$ and $V_0 = Op(K_0)$.

Step-1: $\{G_t\}_{t \in \Delta_1}$ defines a homotopy of sections

$$F_t^1 : V_0 \rightarrow \mathcal{R}_{iso-symp} \subset J^1(V_0, W)$$

such that base of F_t is f for all $t \in [0, 1/N]$. By 2.8 above one can ε_1 -approximate F_t^1 by $J^1 \tilde{f}_t$ over $Op(h_1^1(K_0))$, where h_1^1 is a δ/N -small symplectomorphism and $\tilde{f}_t : V_0 \rightarrow W$ for $t \in [0, 1/N]$ are isosymplectic immersions. Set $K_1 = h_1^1(K_0)$, $V_1 = Op(K_1)$.

Step-2: As ε_1 was chosen small, we can approximate $\{G_t\}_{t \in \Delta_2}$ by $\{G_t^2\}_{t \in \Delta_2}$ such that $\{G_t^2\}_{t \in \Delta_2}$ covers $\tilde{f}_{1/N}$ and $G_{1/N}^2 = G(d\tilde{f}_{1/N})$. So $\{G_t^2\}_{t \in \Delta_2}$ defines a homotopy of sections

$$F_t^2 : V_1 \rightarrow \mathcal{R}_{iso-symp} \subset J^1(V_1, W)$$

such that base of F_t^2 is $\tilde{f}_{1/N}$. Again by 2.8 one can ε_2 -approximate F_t^2 by $J^1 \tilde{f}_t$ over $Op(h_2^1(K_1))$, where h_2^1 is a δ/N -small symplectomorphism and $\tilde{f}_t : V_1 \rightarrow W$ for $t \in [1/N, 2/N]$ are isosymplectic immersions. Set $K_2 = h_2^1(K_1)$, $V_2 = Op(K_2)$. Continue this way till $i = N$. Set $\tilde{K} = K_N$. Now we can define the required f_t in the following way. For $t \in \Delta_i$ set

$$f_t : Op(K_N) \rightarrow W$$

as follows

Take $v \in Op(K_N)$. So there is an unique $v' \in Op(K_i)$ such that

$$h_N^1 \circ \dots \circ h_{i+1}^1(v') = v$$

So set $f_t(v) = \tilde{f}_t(v')$. \square

We now state a theorem which will help us complete the proof of 1.2.

Theorem 2.9. ([1]) *Let (V, ω_V) be an open symplectic manifold and let $K \subset V$ be a core of it, then there exists a homotopy of isosymplectic immersions $g_t : V \rightarrow V$ such that*

$$g_0 = id_V, \text{ and } g_1(V) \subset Op(K)$$

Proof. (Proof of 1.2) Let $K \subset V$ be a core. Use 2.1 to approximate G_t near $\tilde{K} = h^1(K)$ by a homotopy of isosymplectic immersions $\tilde{f}_t : Op_V(\tilde{K}) \rightarrow W$. As $A \subset Sp_{2n}(W)$ is open, a sufficiently close approximation will give us $Gd\tilde{f}_1(Op_V(\tilde{K})) \subset A$. Now by 2.9 there exists a homotopy of isosymplectic immersions g_t with the above properties with respect to the core \tilde{K} . So the required homotopy is $f_t = \tilde{f}_t \circ g_1$. \square

3. GENERALIZATION TO CLOSED MANIFOLDS

In this section we generalize the symplectic A -directed immersion theorem to closed manifolds. We introduce the following notions.

Definition 3.1. Let $n < m \leq q$. An open set $A \subset Sp_{2n}(W)$ is called m -complete if there exists an open set $\hat{A} \subset Sp_{2m}(W)$ such that

$$A = \cup_{\hat{L} \in \hat{A}} Sp_{2n}(\hat{L})$$

Lemma 3.2. Let $Sp_{m,n}(W)$ be the manifold of all $(2m, 2n)$ -symplectic flags on W , i.e.,

$$Sp_{m,n}(W) = \{(\hat{L}, L) : \hat{L} \in Sp_{2m}(W) \text{ and } L \in Sp_{2n}(\hat{L})\}$$

Consider the natural projection $Sp_{m,n}(W) \xrightarrow{P} Sp_{2n}(W)$ and let the following data be given

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & Sp_{m,n}(W) \\ \downarrow & & \downarrow P \\ X \times I & \xrightarrow{F} & Sp_{2n}(W) \end{array}$$

Such that $\pi \circ F = \pi \circ P \circ f$ then F lifts to a map $\tilde{F} : X \times I \rightarrow Sp_{m,n}(W)$, where $\pi : Sp_{2n}(W) \rightarrow W$ be the bundle projection.

Proof. Let $F(x, 0) \subset T_y W$ for some $y \in W$. So $F(x, 0)$ is a symplectic subspace of $T_y W$. Denote its symplectic complement by G_x . Now if F is such that $F(x, t) \cap G_x = \{0\}$, for all t and all x . Then defining $\tilde{F}(x, t) = (F(x, t) \oplus K_x, F(x, t))$ for some $2(m-n)$ -dimensional symplectic subspace K_x of G_x provides a lift.

Now let us consider the general case. Subdivide the parameter interval I into subintervals $\Delta_i = [i/N, (i+1)/N]$ so that on each Δ_i , $F(x, t)$ is δ -small, where δ is a small positive real number. Let us choose $2(m-n)$ -dimensional symplectic subspaces of the symplectic complements of $F(x, i/N)$ and denote it by $G(x, i/N)$ with $f(x, 0) = (F(x, 0) \oplus G(x, 0), F(x, 0))$. Let s be a homeomorphism from Δ_i to I such that $s(i/N) = 0$ and $s((i+1)/N) = 1$. Define for $t \in \Delta_i$

$$G(x, t) = \{(1 - s(t))v + s(t)v', \text{ where } v \in G(x, i/N) \text{ and } v' \in G(x, (i+1)/N)\}$$

Observe that as $F(x, t)$ is δ -small for $t \in \Delta_i$ so is $G(x, t)$ for $t \in \Delta_i$. So for $t \in \Delta_i$, $G(x, t)$ consists of symplectic subspaces only. Now $\tilde{F}(x, t) = (F(x, t) \oplus G(x, t), F(x, t))$ provides the lift. \square

Theorem 3.3. Let $A \subset Sp_{2n}(W)$ be an open set which is m -complete. Then the statement of 1.2 holds for closed V .

Proof. Let $Sp_{m,n}(W)$ be the manifold of all $(2m, 2n)$ -symplectic flags on W . Further let us denote the natural projections by

$$Sp_{m,n}(W) \xrightarrow{\hat{P}} Sp_{2m}(W) \text{ and } Sp_{m,n}(W) \xrightarrow{P} Sp_{2n}(W)$$

Set

$$\bar{A} = \{(\hat{L}, L) : \hat{L} \in \hat{A}, L \in Sp_{2n}(\hat{L})\} \subset Sp_{m,n}(W)$$

In the above \hat{A} exists by m -completeness. Note that $\hat{P}(\bar{A}) = \hat{A}$, $P(\bar{A}) = A$. Let $G_t : V \rightarrow Sp_{2n}(W)$ be the homotopy between the tangential lift $G_0 = Gdf_0$ of the isosymplectic immersion f_0 and the map $G_1 : V \rightarrow A$. Let the map G_1 lifts to a map $\bar{G}_1 : V \rightarrow \bar{A} \subset Sp_{m,n}(W)$. Then the homotopy G_t lifts to a homotopy $\bar{G}_t : V \rightarrow Sp_{m,n}(W)$ by 3.2. We have $G_t = P \circ \bar{G}_t$. Set $\hat{G}_t = \hat{P} \circ \bar{G}_t$, $t \in I$. Let N be the total space of the vector bundle over V whose fiber over a point $v \in V$ is the normal space to $G_1(v)$ in $\hat{G}_1(v)$.

Now extend the isosymplectic immersion f_0 to an immersion $F : Op_N(V) \rightarrow W$ such that $GdF|_V = (\hat{G}_0)|_V$. Consider $F^*(\omega_W)$, observe that $F^*(\omega_W)|_V = \omega_V$. Let $\tilde{\omega}$ be the symplectic structure on $Op_N(V)$ as in (9.3.2) [2]. As $Op_N(V)$ is arbitrarily small we can assume that the linear homotopy ω_t joining $F^*(\omega_W)$ and $\tilde{\omega}$ consist of symplectic structures only. So by symplectic stability theorem (9.3.2) [2] there exists an isotopy $\phi_t : Op_N(V) \rightarrow Op_N(V)$ such that $\phi_t^*(F^*(\omega_W)) = \omega_t$ and $(\phi_t)|_V = id_V$. Set $\tilde{f}_0 = F \circ \phi_1$. So we have constructed an isosymplectic immersion \tilde{f}_0 extending f_0 such that $(Gd\tilde{f}_0)|_V = (\hat{G}_0)|_V$. Hence by 1.2 we can construct $\hat{f}_t : Op_N(V) \rightarrow W$ such that \hat{f}_1 is an \hat{A} -directed immersion. Then $f_t = (\hat{f}_t)|_V$ is the required one.

Now in general G_1 does not lift to \bar{G}_1 . By 2.1 we can assume that the homotopy G_t is constant on a neighborhood $Op(K)$ of the $(2n-1)$ -skeleton of some triangulation of V^{2n} . So we only need to construct f_t on the top simplex Δ of the triangulation keeping f_t fixed on $Op(\partial\Delta)$. If the triangulation is sufficiently small then $(G_1)|_\Delta$ lifts to $\bar{G}_1 : \Delta \rightarrow \bar{A}$ and one can apply the previous argument. \square

4. APPLICATION TO POISSON GEOMETRY

Let $(V^{2n}, \omega_V = d\alpha_V)$ be a symplectic manifold and $(W^{2q}, \omega_W = d\alpha_W)$ be another symplectic manifold such that there is a codimension $2k$ symplectic foliation \mathcal{F} on W with $q > n > k$. By a symplectic foliation we mean that $(\omega_W^{q-k})|_{T\mathcal{F}}$ is nowhere vanishing. Such a W admits a regular Poisson structure. The foliation is known as the characteristic foliation. In this section we study the existence of a regular Poisson structures on a symplectic manifold $(V^{2n}, \omega_V = d\alpha_V)$ with codimension $2k$ characteristic foliation induced from the symplectic structure ω_V .

Theorem 4.1. *Let $(V^{2n}, \omega_V = d\alpha_V)$, $(W^{2q}, \omega_W = d\alpha_W)$ and \mathcal{F} be as above. Set*

$$A = \{L \in Sp_{2n}(W) : L \pitchfork T\mathcal{F}\}$$

Let there exists an isosymplectic immersion $f : V \rightarrow W$ whose tangential lift is homotopic to a map $G_1 : V \rightarrow A$ through a homotopy $G_t : V \rightarrow Sp_{2n}(W)$ such that $\pi \circ G_t = f$. Then a regular Poisson structure on V exists with codimension $2k$ characteristic foliation which is induced from ω_V if either V is open or A is m -complete for some m .

Proof. Observe that A is open. So by 1.2 or by 3.3 there exists an isosymplectic immersion $f_1 : V \rightarrow W$ such that $f_1 \pitchfork \mathcal{F}$. So $f_1^* \mathcal{F}$ is a symplectic foliation on (V, ω_V) . \square

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